Lecture 9
Linear programming
Generalized transportation problem
Sometimes a standard transportation problem (STP) is not sufficient to be a model of a real decision to be made. Necessary extensions may be simple like eg. limited capacities for some routes (new constraints - variables less-or-equal than some numbers) or various prices of the transported commodity at particular sources (the objective function must be then modified – this is production-transportation problem).
However, things can be not that simple. Fixed costs connected with car-based transportation, fixed-size containers or losses of the commodity during transportation make corresponding models much more complicated. Many of the above issues can be addressed as so-called generalized transportation problem (GTP).
Generalized transportation problem is formulated as a problem apparently nearly identical to the standard transportation problem (STP) formulated in the classical notation. The formulation uses double indexes for all the variables. There are also variables non-negativity constraints and two groups of linear constraints (equalities or inequalities) in which variables are grouped with respect to either first or the second index. The main difference to compare GTP with STP is that one of the groups of the abovementioned constraints is composed not of sums of variables but of sums of products of parameters and variables.
The generalized transportation problem deals with a network of \( m \) sources (or suppliers) which are capable of supplying some uniform commodity to \( n \) destinations where the commodity is demanded/needed. Sources and destinations are connected with so-called routes. A unit cost (cost of transporting/shipping one unit of commodity from a source to the destination) is assigned to each route.

Moreover, there are additional parameters assigned to each route which reflect changes of the amount of commodity transported via this route.
Generalized transportation problem – definition (2)

The goal is to minimize the overall transportation (or shipping) cost of the commodity on all the routes while using capacities of sources/suppliers and satisfying demands of destinations.

What to calculate?

It is necessary to plan how much commodity should be shipped over each possible route in order to minimize the total shipping cost over all the routes while using capacities of sources (suppliers), satisfying demands of destinations and taking into account the changes of amounts of the commodity on each route.
Generalized transportation problem - parameters (1)

The following parameters are given:

- \( a_i, i = 1, \ldots, m \) – capacities of sources/suppliers (maximal possible supplies) – amount of the commodity stored (or possible to be stored/produced over some specific period of time) at the \( i \)-th source/supplier (measured in t, kg, m\(^2\), m\(^3\), m, pieces, etc.);

- \( b_j, j = 1, \ldots, n \) – demands of destinations - amount of the commodity demanded/needed by the \( j \)-th destination (over some specific period of time) (measured in the same unit as supplier capabilities);
Generalized transportation problem - parameters (2)

\[ c_{ij}, \ i = 1, \ldots, m; \ j = 1, \ldots, n \] – a unit cost i.e. a cost of transporting one unit of the commodity of from the \( i \)-th source/supplier to the \( j \)-th destination i.e. on the \( i-j \) route (measured in €/t, €/kg, €/m\(^2\), €/m\(^3\), €/piece) etc. – the unit of the commodity must be same as for \( a_i \) and \( b_j \), the currency unit can be arbitrary but obviously the same for all the routes);
Generalized transportation problem - parameters (3)

\( p_{ij}, i = 1, \ldots, m, j = 1, \ldots, n \) – a preservation coefficient - a unitless number describing the change of the amount of the commodity transported from the \( i \)-th source/supplier to the \( j \)-th destination i.e. on the \( i - j \) route. In the most common case, it is a coefficient from the interval \([0,1]\) which means the loss of the commodity i.e. what fraction of the commodity which enters the route \( i - j \) leaves it:

- 0 means no commodity can be transported (forbidden route);
- 1 means no loss of the commodity;
- any value between 0 and 1 means the commodity is partially lost during the transportation process.

Other interpretations of coefficients \( p_{ij} \) are also possible (and they do not need to be unitless).
Issues of capacity-demand balance

As in the STP, in GTP the total source/supplier capacities and total demand requirements can be either equal or different. However, the equality variant $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ does not mean that all the balance constraints are equality constraints because of loss (or, in some cases, increase) of amounts of some commodities.

There may be two variants of the problem with slightly complicated formulation than in case of ST:

1) $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$ total source/supplier capacities are larger than total destination demands

2) $\sum_{i=1}^{m} a_i \leq \sum_{j=1}^{n} b_j$ total source/supplier capacities are smaller or equal than total destination demands.
Generalized transportation problem –
general mathematical model (1)

Decision variables
\( x_{ij}, \ i = 1, \ldots, m, j = 1, \ldots, n \) - amount of the commodity transported from the \( i \)-th source/supplier to the \( j \)-th destination (the total number of the variables is \( m \cdot n \))

\[
\begin{align*}
&c_{11} x_{11} + c_{12} x_{12} + \ldots + c_{1n} x_{1n} + \\&c_{21} x_{21} + c_{22} x_{22} + \ldots + c_{2n} x_{2n} + \\&\ldots + \\&c_{m1} x_{m1} + c_{m2} x_{m2} + \ldots + c_{mn} x_{mn} \rightarrow \min
\end{align*}
\]

- total shipping cost of the commodity through all the routes

subject to the constraints
\( x_{ij} \geq 0, \ i = 1, \ldots, m, \ j = 1, \ldots, n \) - amounts of the commodity cannot be negative
## Generalized transportation problem – general mathematical model (2a)

1. Total source/supplier capacities are larger than total destination demands

\[
x_{11} + x_{12} + \ldots + x_{1n} \leq a_1 \\
x_{21} + x_{22} + \ldots + x_{2n} \leq a_2 \\
\vdots \\
x_{m1} + x_{m2} + \ldots + x_{mn} \leq a_m
\]

**Balance constraints for sources/suppliers**

(The capacity of at least one source/supplier is not fully used - at least one source/supplier does not send all the commodity it can)

\[
p_{11}x_{11} + p_{21}x_{21} + \ldots + p_{m1}x_{m1} = b_1 \\
p_{12}x_{12} + p_{22}x_{22} + \ldots + p_{m2}x_{m2} = b_2 \\
\vdots \\
p_{1n}x_{1n} + p_{2n}x_{2n} + \ldots + p_{mn}x_{mn} = b_n
\]

**Balance constraints for destinations**

(All the destinations have their demands satisfied regarding the losses of the commodity expressed by \(p_{ij}\) – they receive all the commodity they need)
Generalized transportation problem – general mathematical model (2b)

2) Total source/supplier capacities are smaller or equal than total destination demands

<table>
<thead>
<tr>
<th>Source/supplier constraints</th>
<th>Destination constraints</th>
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| $x_{11} + x_{12} + \ldots + x_{1n} = a_1$ | Balance constraints for sources/suppliers
| $x_{21} + x_{22} + \ldots + x_{2n} = a_2$ | (the capacities of all the sources/suppliers are fully used - they send all the commodities they can)
| $\vdots$ | |
| $x_{m1} + x_{m2} + \ldots + x_{mn} = a_m$ | |
| $p_{11}x_{11} + p_{21}x_{21} + \ldots + p_{m1}x_{m1} \leq b_1$ | Balance constraints for destinations
| $p_{12}x_{12} + p_{22}x_{22} + \ldots + p_{m2}x_{m2} \leq b_2$ | (the demand of at least one destination is not fully satisfied regarding the losses of the commodity expressed by $p_{ij}$ – at least one destination does not receive all the commodities it needs)
| $\vdots$ | |
| $p_{1n}x_{1n} + p_{2n}x_{2n} + \ldots + p_{mn}x_{mn} \leq b_n$ | |
1. GTP, unlike STP, does not guarantee the existence of an optimal solution because the constraints may be contradictory.

   This may happen with variant 1 if the total capacity of sources/suppliers, whereas larger than the total demand of capacities is not enough to „cover” losses of the commodity on the routes. Even if all the sources/suppliers send all the commodities they can to the destinations, too much is lost to satisfy the demands.

   In this case the problem must be reformulated to the variant 2 (not all the destinations obtain what they need).

2. GTP does not guarantee the existence of an integer optimal solution as STP does (under simple assumptions on some parameters being integer).
3. GTP, unlike STP, cannot be reformulated to the unified notation. Because of losses/increases of amounts of the commodity, it is hard to determine what the total amount of the commodity really is and that’s why it is impossible to add the constraint „the total amount of the commodity transported via all the routes is equal to some specific amount”. 
Generalized transportation problem – remarks (2)

4. Models of other problems based on the mathematical formulation of GTP may slightly differ from the formulation considered in the lecture. This includes sums of products in the first group of balance constraints instead of in the second one, the presence of $\geq$ constraints instead of $=$ constraints, the integer constraints on variables.

Such formulations are used eg. to create advanced versions of the assignment problems (allocation of production orders among various production units etc.)